

# THE REIDEMEISTER ZETA FUNCTION FOR ALMOST BIEBERBACH GROUPS

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**ABSTRACT.** We show that Reidemeister zeta function is a rational function for almost Bieberbach groups, when the corresponding Nielsen number equals to the absolute value of the Lefschetz number. Using this result we are able to write the functional equation for the Reidemeister zeta function and find a connection between the Reidemeister zeta function and the Reidemeister torsion of the corresponding mapping torus.

## 0. INTRODUCTION

We assume everywhere  $X$  to be a connected, compact polyhedron and  $f : X \rightarrow X$  to be a continuous map. Let  $p : \tilde{X} \rightarrow X$  be the universal cover of  $X$  and  $\tilde{f} : \tilde{X} \rightarrow \tilde{X}$  a lifting of  $f$ , i.e.,  $p \circ \tilde{f} = f \circ p$ . Two liftings  $\tilde{f}$  and  $\tilde{f}'$  are called *conjugate* if there is a  $\gamma \in \Gamma \cong \pi_1(X)$  such that  $\tilde{f}' = \gamma \circ \tilde{f} \circ \gamma^{-1}$ . The subset  $p(\text{Fix}(\tilde{f})) \subset \text{Fix}(f)$  is called the *fixed point class* of  $f$  determined by the lifting class  $[\tilde{f}]$ . A fixed point class is called *essential* if its index is nonzero. The number of lifting classes of  $f$  (and hence the number of fixed point classes, empty or not) is called the *Reidemeister number* of  $f$ , denoted by  $R(f)$ . This is a positive integer or infinity. The number of essential fixed point classes is called the *Nielsen number* of  $f$ , denoted by  $N(f)$  [19].

The Nielsen number is always finite.  $R(f)$  and  $N(f)$  are homotopy invariants. In the category of compact, connected polyhedra the Nielsen number of a map is, apart from in certain exceptional cases, equal to the least number of fixed points of maps with the same homotopy type as  $f$ .

Let  $G$  be a group and  $\phi : G \rightarrow G$  an endomorphism. Two elements  $\alpha, \alpha' \in G$  are said to be  $\phi$ -conjugate if and only if there exists  $\gamma \in G$  such that  $\alpha' = \gamma \alpha \phi(\gamma)^{-1}$ . The number of  $\phi$ -conjugacy classes is called the *Reidemeister number* of  $\phi$ , denoted by  $R(\phi)$ .

Taking a dynamical point of view, we consider the iterates of  $f$  and  $\phi$ , and we may define several zeta functions connected with the Nielsen fixed point theory following [6, 30, 7, 8]). **We assume throughout this article that  $R(f^n) < \infty$  and  $R(\phi^n) < \infty$  for all  $n > 0$ .** The Reidemeister zeta

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functions of  $f$  and  $\phi$  and the Nielsen zeta function of  $f$  are defined as power series:

$$\begin{aligned} R_\phi(z) &= \exp \left( \sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n \right), \\ R_f(z) &= \exp \left( \sum_{n=1}^{\infty} \frac{R(f^n)}{n} z^n \right), \\ N_f(z) &= \exp \left( \sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n \right). \end{aligned}$$

$R_f(z)$  and  $N_f(z)$  are homotopy invariants. The function  $N_f(z)$  has a positive radius of convergence which has a sharp estimate in terms of the topological entropy of the map  $f$  [30]. The above zeta functions are directly analogous to the Lefschetz zeta function

$$L_f(z) := \exp \left( \sum_{n=1}^{\infty} \frac{L(f^n)}{n} z^n \right),$$

where

$$(1) \quad L(f^n) := \sum_{k=0}^{\dim X} (-1)^k \operatorname{tr} \left[ f_{*k}^n : H_k(X; \mathbb{Q}) \rightarrow H_k(X; \mathbb{Q}) \right]$$

is the Lefschetz number of the iterate  $f^n$  of  $f$ . The Lefschetz zeta function is a rational function of  $z$  and is given by the formula:

$$L_f(z) = \prod_{k=0}^{\dim X} \det (I - f_{*k} \cdot z)^{(-1)^{k+1}}.$$

A. Weil [33] introduced the function  $L_f(z)$  in his study of the fixed points of the Frobenius map of an algebraic variety defined over a finite field. In the theory of discrete dynamical systems  $L_f(z)$  was defined by Smale [32] and in the singularity theory by Milnor [28].

The following problem was investigated: for which spaces and maps and for which groups and homomorphisms are the Nielsen and Reidemeister zeta functions rational functions? Are these functions algebraic functions?

The knowledge that a zeta function is rational is important because it shows that the infinite sequence of coefficients of the corresponding power series is closely interconnected, and is given by the finite set of zeros and poles of the zeta function.

In [8, 10, 11, 25, 9], the rationality of the Reidemeister zeta function was proven in the following cases: the group is finitely generated and an endomorphism is eventually commutative; the group is finite; the group is a direct sum of a finite group and a finitely generated free Abelian group; the group is finitely generated, nilpotent and torsion free.

In [34, Theorem 4] the rationality of the Reidemeister and Nielsen zeta function was proven for infra-nilmanifold under some (rather technical) sufficient conditions.

It is also known that the Reidemeister numbers of the iterates of an automorphism of almost Bieberbach group satisfy remarkable Gauss congruences [12, 13].

In this paper we investigate the Reidemeister zeta function for almost Bieberbach groups. The main result (Theorem 1.6) of this paper is that the Reidemeister zeta function of any endomorphism of any almost Bieberbach group is a rational function under the additional condition that  $N(f) = |L(f)|$ . Using this result we are able to write the functional equation for the Reidemeister zeta function (Theorem 2.2) and to find a connection between the Reidemeister zeta function and the Reidemeister torsion of the corresponding mapping torus (Theorem 3.2).

During the preparation of this paper, the authors became aware of the paper [2] by Karel Dekimpe and Gert-Jan Dugardein, on the same subject. Theorem 1.6 of the present paper is also proven in [2, Theorem 4.4].

The paper [2] claims that the Nielsen zeta function of any map on any infra-nilmanifold is a rational function.

## 1. THE REIDEMEISTER ZETA FUNCTION FOR ALMOST BIEBERBACH GROUPS

In this section we consider almost Bieberbach groups  $\pi \subset G \rtimes \text{Aut}(G)$ , where  $G$  is a connected, simply connected nilpotent Lie group, and infra-nilmanifolds  $\pi \backslash G$ . It is known that these are exactly the class of almost flat Riemannian manifolds. It is L. Auslander's result (see, for example, [24]) that  $\Gamma := \pi \cap G$  is a lattice of  $G$ , and is the unique maximal normal nilpotent subgroup of  $\pi$ . The group  $\Phi = \pi/\Gamma$  is the *holonomy group* of  $\pi$  or infra-nilmanifold  $\pi \backslash G$ . It sits naturally in  $\text{Aut}(G)$ .

Let  $M = \pi \backslash G$ . Any continuous map  $f : M \rightarrow M$  induces a homomorphism  $\phi : \pi \rightarrow \pi$ . Due to [23, Theorem 1.1], we can choose an affine element  $(d, D) \in G \rtimes \text{Endo}(G)$  such that

$$(2) \quad \phi(\alpha) \circ (d, D) = (d, D) \circ \alpha, \quad \forall \alpha \in \pi.$$

This implies that the affine map  $(d, D) : G \rightarrow G$  induces a continuous map on the infra-nilmanifold  $M = \pi \backslash G$ .

Let  $A \in \Phi$ . Then we can choose  $g \in G$  so that  $\alpha = (g, A) \in \pi$ . Write  $\phi(\alpha) = (g', A')$ . By (2), we have  $(g', A')(d, D) = (d, D)(g, A) \Rightarrow A'D = DA$ . Thus  $\phi$  induces a function  $\hat{\phi} : \Phi \rightarrow \Phi$  given by  $\hat{\phi}(A) = A'$  so that it satisfies that

$$\hat{\phi}(A)D = DA$$

for all  $A \in \Phi$ .

By [22, Lemma 3.1], we can choose a fully invariant subgroup  $\Lambda \subset \Gamma$  of  $\pi$  which is of finite index. Therefore  $\phi(\Lambda) \subset \Lambda$  and so  $\phi$  induces the following

commutative diagram

$$\begin{array}{ccccccc}
1 & \longrightarrow & \Lambda & \xrightarrow{j} & \pi & \xrightarrow{q} & \Psi \longrightarrow 1 \\
& & \downarrow \phi' & & \downarrow \phi & & \downarrow \bar{\phi} \\
1 & \longrightarrow & \Lambda & \xrightarrow{j} & \pi & \xrightarrow{q} & \Psi \longrightarrow 1
\end{array}$$

where  $\Psi = \pi/\Lambda$ . Applying (2) for  $\lambda \in \Lambda \subset \pi$ , we see that

$$\phi(\lambda) = dD(\lambda)d^{-1} = (\tau_d D)(\lambda)$$

where  $\tau_d$  is the conjugation by  $d$ . The homomorphism  $\phi' : \Lambda \rightarrow \Lambda$  induces a unique Lie group homomorphism  $F = \tau_d D : G \rightarrow G$ , and hence a Lie algebra homomorphism  $F_* : \mathfrak{G} \rightarrow \mathfrak{G}$ . On the other hand, since  $\phi(\Lambda) \subset \Lambda$ ,  $f$  has a lift  $\bar{f} : N \rightarrow N$  on the nilmanifold  $N := \Lambda \backslash G$  which finitely and regularly covers  $M$  and has  $\Psi$  as the group of covering transformations.

**Theorem 1.1** ([17, Theorem 4.2]). *We have*

$$R(f) = \frac{1}{|\Psi|} \sum_{\bar{\alpha} \in \Psi} R(\bar{\alpha} \bar{f}) = \frac{1}{|\Psi|} \sum_{\bar{\alpha} \in \Psi} \sigma(L(\bar{\alpha} \bar{f})),$$

where  $\sigma : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$  is defined by  $\sigma(0) = \infty$  and  $\sigma(x) = |x|$  for all  $x \neq 0$ .

Furthermore, we can rewrite this averaging formula in terms of the holonomy group of the infra-nilmanifold.

**Theorem 1.2** ([22, Theorem 3.4], [17, Theorem 6.11]). *We have*

$$\begin{aligned}
L(f) &= \frac{1}{|\Phi|} \sum_{A \in \Phi} \frac{\det(A_* - F_*)}{\det A_*} = \frac{1}{|\Phi|} \sum_{A \in \Phi} \det(I - A_* F_*), \\
N(f) &= \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(A_* - F_*)| = \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(I - A_* F_*)|, \\
R(f) &= R(\phi) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \sigma(\det(A_* - F_*)).
\end{aligned}$$

In the above theorem, instead of  $F_* = (\tau_d D)_* = \text{Ad}(d)D_*$  we may use  $D_*$  because

$$\begin{aligned}
\det(I - A_* F_*) &= \det(A_*^{-1}(I - A_* \text{Ad}(d)D_*)A_*) \\
&= \det(I - \text{Ad}(d)D_* A_*) = \det(I - D_* A_*) \\
&= \det(A_*(I - D_* A_*)A_*^{-1}) = \det(I - A_* D_*).
\end{aligned}$$

**1.1. Topological entropy and radius of convergence of the Reide-meister zeta function.** The most widely used measure for the complexity of a dynamical system is the topological entropy. For the convenience of the reader, we include its definition. Let  $f : X \rightarrow X$  be a self-map of a compact metric space. For given  $\epsilon > 0$  and  $n \in \mathbb{N}$ , a subset  $E \subset X$  is said to be  $(n, \epsilon)$ -separated under  $f$  if for each pair  $x \neq y$  in  $E$  there is  $0 \leq i < n$  such that  $d(f^i(x), f^i(y)) > \epsilon$ . Let  $s_n(\epsilon, f)$  denote the largest cardinality of

any  $(n, \epsilon)$ -separated subset  $E$  under  $f$ . Thus  $s_n(\epsilon, f)$  is the greatest number of orbit segments  $x, f(x), \dots, f^{n-1}(x)$  of length  $n$  that can be distinguished one from another provided we can only distinguish between points of  $X$  that are at least  $\epsilon$  apart. Now let

$$h(f, \epsilon) := \limsup_n \frac{1}{n} \cdot \log s_n(\epsilon, f)$$

$$h(f) := \limsup_{\epsilon \rightarrow 0} h(f, \epsilon).$$

The number  $0 \leq h(f) \leq \infty$ , which to be independent of the metric  $d$  used, is called the topological entropy of  $f$ . If  $h(f, \epsilon) > 0$  then, up to resolution  $\epsilon > 0$ , the number  $s_n(\epsilon, f)$  of distinguishable orbit segments of length  $n$  grows exponentially with  $n$ . So  $h(f)$  measures the growth rate in  $n$  of the number of orbit segments of length  $n$  with arbitrarily fine resolution.

A basic relation between the topological entropy  $h(f)$  and the Nielsen numbers  $N(f^n)$  was found by N. Ivanov [18]. We present here a very short proof by Boju Jiang of the Ivanov's inequality.

**Lemma 1.3** ([18]). *Let  $f$  be a continuous map of a compact connected polyhedron  $X$  into itself. Then*

$$h(f) \geq \limsup_n \frac{1}{n} \cdot \log N(f^n)$$

*Proof.* Let  $\delta$  be such that every loop in  $X$  of diameter  $< 2\delta$  is contractible. Let  $\epsilon > 0$  be a smaller number such that  $d(f(x), f(y)) < \delta$  whenever  $d(x, y) < 2\epsilon$ . Let  $E_n \subset X$  be a set consisting of one point from each essential fixed point class of  $f^n$ . Thus  $|E_n| = N(f^n)$ . By the definition of  $h(f)$ , it suffices to show that  $E_n$  is  $(n, \epsilon)$ -separated. Suppose it is not so. Then there would be two points  $x \neq y \in E_n$  such that  $d(f^i(x), f^i(y)) \leq \epsilon$  for  $0 \leq i < n$  hence for all  $i \geq 0$ . Pick a path  $c_i$  from  $f^i(x)$  to  $f^i(y)$  of diameter  $< 2\epsilon$  for  $0 \leq i < n$  and let  $c_n = c_0$ . By the choice of  $\delta$  and  $\epsilon$ ,  $f \circ c_i \simeq c_{i+1}$  for all  $i$ , so  $f^n \circ c_0 \simeq c_n = c_0$ . This means  $x, y$  in the same fixed point class of  $f^n$ , contradicting the construction of  $E_n$ .  $\square$

This inequality is remarkable in that it does not require smoothness of the map and provides a common lower bound for the topological entropy of all maps in a homotopy class.

Recall that we are assuming that  $R(f^n) = R(\phi^n)$  is finite for all  $n$ . By [17, Theorem 4.1], it is equivalent to saying that  $\det(A_* - F_*^n) \neq 0$  for all  $A \in \Phi$  and all  $n$ , and hence  $\sigma(\det(A_* - F_*^n)) = |\det(A_* - F_*^n)|$ . Thus

$$\begin{aligned} R(f^n) &= R(\phi^n) = \frac{1}{|\Phi|} \sum_{A \in \Phi} \sigma(\det(A_* - F_*^n)) \\ &= \frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(A_* - F_*^n)| = N(f^n). \end{aligned}$$

We denote by  $R$  the radius of convergence of the Reidemeister zeta function  $R_f(z) = R_\phi(z)$ . Let  $h = \inf h(g)$  over all maps  $g$  which are homotopic to  $f$ .

**Theorem 1.4.** *Let  $f$  be a continuous map  $f$  of an infra-nilmanifold  $M$  into itself. The Reidemeister zeta function  $R_f(z) = R_\phi(z)$  has positive radius of convergence  $R$  which admits following estimation*

$$R \geq \exp(-h) > 0,$$

where  $h = \inf h(g)$  over all maps  $g$  which are homotopic to  $f$ .

*Proof.* The inequality  $R \geq \exp(-h)$  follows from the equality  $R(f^n) = R(\phi^n) = N(f^n)$  for all  $n$ , Lemma 1.3, the Cauchy-Hadamard formula, and the homotopy invariance of the radius  $R$  of the Reidemeister zeta function  $R_f(z)$ . We consider a smooth map  $g : M \rightarrow M$  which is homotopic to  $f$ . As is known [31], the entropy  $h(g)$  is finite. Thus  $\exp(-h) \geq \exp(-h(g)) > 0$ .  $\square$

## 1.2. Rationality of Reidemeister zeta function.

**Lemma 1.5.** *For an almost Bieberbach group with the holonomy group  $\Phi$ , the Reidemeister zeta function is*

$$R_\phi(z) = \prod_{A \in \Phi} \sqrt[|\Phi|]{\exp \left( \sum_{n=1}^{\infty} \frac{|\det(A_* - F_*^n)|}{n} z^n \right)}.$$

*Proof.* We have

$$\begin{aligned} N_f(z) = R_\phi(z) &= \exp \left( \sum_{n=1}^{\infty} \frac{R(\phi^n)}{n} z^n \right) \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{\frac{1}{|\Phi|} \sum_{A \in \Phi} |\det(A_* - F_*^n)|}{n} z^n \right) \\ &= \prod_{A \in \Phi} \left( \exp \left( \sum_{n=1}^{\infty} \frac{|\det(A_* - F_*^n)|}{n} z^n \right) \right)^{\frac{1}{|\Phi|}} \\ &= \prod_{A \in \Phi} \sqrt[|\Phi|]{\exp \left( \sum_{n=1}^{\infty} \frac{|\det(A_* - F_*^n)|}{n} z^n \right)}. \end{aligned} \quad \square$$

**Theorem 1.6.** *Let  $f : M \rightarrow M$  be a continuous map of an infra-nilmanifold  $M$ . Assume  $N(f) = |L(f)|$ . Then the Reidemeister zeta function  $R_f(z) = R_\phi(z)$  is a rational function and is equal to*

$$R_f(z) = R_\phi(z) = N_f(z) = L_f((-1)^p z)^{(-1)^r}$$

where  $p$  is the number of real eigenvalues of  $F_*$  in the region  $(-\infty, -1)$  and  $r$  is the number of real eigenvalues of  $F_*$  whose absolute value is greater than 1.

*Proof.* We are assuming that  $R(f^n) = R(\phi^n)$  is finite for all  $n$ . By [17, Theorem 4.1], it is equivalent to saying that  $\det(A_* - F_*^n) \neq 0$  for all  $A \in \Phi$  and all  $n$ . Thus  $R(f^n) = R(\phi^n) = N(f^n)$  for all  $n$ .

By Theorem 8.2.2 [29], page 127,  $N(f) = |L(f)|$  implies  $N(f^n) = |L(f^n)|$  for all  $n$ . Let  $\epsilon_n$  be the sign of  $\det(I - F_*^n)$ . As before, let  $p$  be the number of real eigenvalues of  $F_*$  which are less than  $-1$  and  $r$  be the number of real eigenvalues of  $F_*$  of modulus  $> 1$ . Then  $\epsilon_n = (-1)^{r+pn}$ . By Theorem 1.2, we have that  $\epsilon_1 \det(I - A_* F_*) \geq 0$  for all  $A \in \Phi$ . In particular, we have

$$\det(I - A_* F_*) \det(I - B_* F_*) \geq 0 \quad \text{for all } A, B \in \Phi.$$

Choose arbitrary  $n > 0$ . By [29, Lemma 8.2.1],

$$\det(I - A_* F_*^n) \det(I - F_*^n) \geq 0 \quad \text{for all } A \in \Phi.$$

Hence we have  $N(f^n) = \epsilon_n L(f^n) = (-1)^{r+pn} L(f^n)$ . Consequently,

$$\begin{aligned} R_\phi(z) = N_f(z) &= \exp \left( \sum_{n=1}^{\infty} \frac{N(f^n)}{n} z^n \right) \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^{r+pn} L(f^n)}{n} z^n \right) \\ &= \left( \exp \left( \sum_{n=1}^{\infty} \frac{L(f^n)}{n} ((-1)^p z)^n \right) \right)^{(-1)^r} \\ &= L_f((-1)^p z)^{(-1)^r} \end{aligned}$$

is a rational function. □

**Remark 1.7.** Anosov [1] proved that  $N(f) = |L(f)|$  for all continuous maps  $f : M \rightarrow M$  if  $M$  is a nilmanifold but he also observed that there is a continuous map  $f : K \rightarrow K$  of the Klein bottle  $K$  (which is a flat manifold having  $\mathbb{Z}_2$  as holonomy group) such that  $N(f) \neq |L(f)|$ . Since then, many generalizations have been obtained. For instance, first the Anosov relation  $N(f) = |L(f)|$  holds for some classes of maps on infra-nilmanifolds:

- the class of homotopically periodic maps [21],
- the class of virtually unipotent maps [26],
- the class of nowhere expanding maps [4].

Secondly the Anosov relation  $N(f) = |L(f)|$  holds for all maps on some classes of infra-nilmanifolds:

- the class of nilmanifolds [1],
- the class of orientable generalized Hantzsche-Wendt manifolds [3],
- the class of infra-nilmanifolds with holonomy group of odd order [4],
- the class of infra-nilmanifolds with a 2-perfect holonomy group [5].

Therefore, for those classes of maps on any infra-nilmanifold or for those classes of infra-nilmanifolds, the Nielsen zeta functions and the Reidemeister zeta functions are rational.

**1.3. Asymptotic Nielsen number.** The growth rate of a sequence  $a_n$  of complex numbers is defined by

$$\text{Growth}(a_n) := \max \left\{ 1, \limsup_{n \rightarrow \infty} |a_n|^{1/n} \right\}.$$

We define the asymptotic Nielsen number [18] and the asymptotic Reidemeister number to be the growth rate  $N^\infty(f) := \text{Growth}(N(f^n))$  and  $R^\infty(f) := \text{Growth}(R(f^n))$ , when all  $R(f^n)$  are finite, correspondingly. These asymptotic numbers are homotopy type invariants. We denote by  $\text{sp}(A)$  the spectral radius of the matrix or the operator  $A$ ,  $\text{sp}(A) = \lim_n \sqrt[n]{\|A^n\|}$  which coincide with the largest modulus of an eigenvalue of  $A$ . We denote by  $\bigwedge F_* := \bigoplus_{\ell=0}^m \bigwedge^\ell F_*$  a linear operator induced in the exterior algebra  $\bigwedge^* \mathbb{R}^m := \bigoplus_{\ell=0}^m \bigwedge^\ell \mathbb{R}^m$  of  $\mathfrak{G}$  considered as the linear space  $\mathbb{R}^m$ .

**Lemma 1.8.** [27] *For a continuous map  $f$  of a nilmanifold,*

$$N^\infty(f) = \text{sp}(\bigwedge F_*)$$

*provided that 1 is not in the spectrum of  $\bigwedge F_*$  and  $\text{sp}(F_*) > 1$ .*

*Proof.* Let  $\{\lambda_1, \dots, \lambda_m\}$  be the eigenvalues of  $F_*$ , counted with multiplicities. First we note from definition that

$$\text{sp}(\bigwedge F_*) = \begin{cases} \prod_{|\lambda_j| > 1} |\lambda_j| & \text{when } \text{sp}(F_*) > 1 \\ \text{sp}(F_*) & \text{when } \text{sp}(F_*) \leq 1. \end{cases}$$

Since  $N(f^n) = |\det(I - F_*^n)| = \prod_{j=1}^m |1 - \lambda_j^n|$ , we have

$$\begin{aligned} \log \limsup_{n \rightarrow \infty} N(f^n)^{1/n} &= \limsup_{n \rightarrow \infty} \frac{1}{n} \sum_{j=1}^m \log |1 - \lambda_j^n| \\ &= \sum_{j=1}^m \limsup_{n \rightarrow \infty} \frac{1}{n} \log |1 - \lambda_j^n|. \end{aligned}$$

If  $|\lambda| \leq 1$  and  $\lambda \neq 1$  then  $\limsup_n \frac{1}{n} \log |1 - \lambda^n| = 0$ . For,  $\log |1 - \lambda^n| \leq \log 2$ . If  $|\lambda| > 1$  then using L'Hôpital's rule, we have  $\lim_{n \rightarrow \infty} \frac{1}{n} \log |1 - \lambda^n| = \log |\lambda|$ . Hence

$$\begin{aligned} N^\infty(f) &= \max \left\{ 1, \limsup_{n \rightarrow \infty} N(f^n)^{1/n} \right\} \\ &= \max \left\{ 1, \prod_{|\lambda| > 1} |\lambda| \right\} = \prod_{|\lambda| > 1} |\lambda| = \text{sp}(\bigwedge F_*). \quad \square \end{aligned}$$

By the Cauchy-Hadamard formula this result gives immediately algebraic formula for the radius of convergence of the Reidemeister and the Nielsen zeta functions.

Using Theorem 1.2, we hope to obtain the same result for all maps on infra-nilmanifolds.



## 2. FUNCTIONAL EQUATION FOR REIDEMEISTER ZETA FUNCTION

## 2.1. Functional equation for Lefschetz zeta function.

**Lemma 2.1** (Functional equation for the Lefschetz zeta function, [14, Proposition 8]). *Let  $M$  be a closed orientable manifold and let  $f : M \rightarrow M$  be a continuous map of degree  $d$ . Then*

$$L_f\left(\frac{\alpha}{dz}\right) = \epsilon (-\alpha dz)^{(-1)^m \chi(M)} L_f(\alpha z)^{(-1)^m}$$

where  $m = \dim M$ ,  $\alpha = \pm 1$  and  $\epsilon \in \mathbb{C}$  is a non-zero constant such that if  $|d| = 1$  then  $\epsilon = \pm 1$ .

*Proof.* In the Lefschetz zeta function formula (1), we may replace  $f_*$  by  $f^* : H^*(M; \mathbb{Q}) \rightarrow H^*(M; \mathbb{Q})$ . Let  $\beta_k = \dim H_k(M; \mathbb{Q})$  be the  $k$ th Betti number of  $M$ . Let  $\lambda_{k,j}$  be the (complex and distinct) eigenvalues of  $f_{*k} : H_k(M; \mathbb{Q}) \rightarrow H_k(M; \mathbb{Q})$ .

Via the natural non-singular pairing in the cohomology  $H^k(M; \mathbb{Q}) \otimes H^{m-k}(M; \mathbb{Q}) \rightarrow \mathbb{Q}$ , the operators  $f_{m-k}^*$  and  $d(f_k^*)$  are adjoint to each other. Hence since  $\lambda_{k,j}$  is an eigenvalue of  $f_k^*$ ,  $\mu_{\ell,j} = d/\lambda_{k,j}$  is an eigenvalue of  $f_{m-k}^* = f_\ell^*$ . Furthermore,  $\beta_k = \beta_{m-k} = \beta_\ell$ .

Consequently, we have

$$\begin{aligned} L_f\left(\frac{\alpha}{dz}\right) &= \prod_{k=0}^m \prod_{j=1}^{\beta_k} \left(1 - \lambda_{k,j} \frac{\alpha}{dz}\right)^{(-1)^{k+1}} \\ &= \prod_{k=0}^m \prod_{j=1}^{\beta_k} \left(1 - \frac{d}{\lambda_{k,j}} \alpha z\right)^{(-1)^{k+1}} \left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^k} \\ &= \prod_{\ell=0}^m \prod_{j=1}^{\beta_{m-\ell}} (1 - \mu_{\ell,j} \alpha z)^{(-1)^{m-\ell+1}} \prod_{k=0}^m \prod_{j=1}^{\beta_k} \left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^{m-\ell}} \\ &= \left( \prod_{\ell=0}^m \prod_{j=1}^{\beta_\ell} (1 - \mu_{\ell,j} \alpha z)^{(-1)^{\ell+1}} \prod_{k=0}^m \prod_{j=1}^{\beta_k} \left(-\frac{\alpha dz}{\lambda_{k,j}}\right)^{(-1)^\ell} \right)^{(-1)^m} \\ &= L_f(\alpha z)^{(-1)^m} \cdot (-\alpha dz)^{\sum_{\ell=0}^m (-1)^\ell \beta_\ell} \cdot \prod_{k=0}^m \prod_{j=1}^{\beta_k} \lambda_{k,j}^{(-1)^{k+1}} \\ &= L_f(\alpha z)^{(-1)^m} \epsilon (-\alpha dz)^{(-1)^m \chi(M)}. \end{aligned}$$

Here,

$$\epsilon = \prod_{k=0}^m \prod_{j=1}^{\beta_k} \lambda_{k,j}^{(-1)^{k+1}} = \pm \prod_{k=0}^m \det(f_k^*).$$

□

**2.2. Functional equation for Reidemeister zeta function.** Recall that we are assuming that  $R(f^n)$  is finite for all  $n$ . Thus  $R(f^n) = R(\phi^n) = N(f^n)$  for all  $n$ . Therefore, the Reidemeister zeta function  $R_f(z)$  coincides with the Nielsen zeta function  $N_f(z)$ .

**Theorem 2.2.** *Let  $f : M \rightarrow M$  be a continuous map of an orientable infra-nilmanifold  $M$ . Assume that  $N(f) = |L(f)|$ . The Reidemeister zeta function  $R_f(z)$  has the following functional equation:*

$$R_f\left(\frac{1}{dz}\right) = (R_f(z))^{(-1)^m} \epsilon (\sigma dz)^{(-1)^r \chi(M)},$$

where  $d$  is a degree  $f$ ,  $m = \dim M$ ,  $\epsilon$  is a constant in  $\mathbb{C}^\times$ ,  $\sigma = (-1)^p$ ,  $p$  is the number of real eigenvalues of  $F_*$  in the region  $(-\infty, -1)$  and  $r$  is the number of real eigenvalues of  $F_*$  whose absolute value is greater than 1. If  $|d| = 1$  then  $\epsilon = \pm 1$ .

*Proof.* From Theorem 1.6 it follows that  $R_f(z) = L_f(\sigma z)^{(-1)^r}$ . Now by Lemma 2.1, we have

$$\begin{aligned} R_f\left(\frac{1}{dz}\right) &= L_f\left(\frac{\sigma}{dz}\right)^{(-1)^r} \\ &= \left(\epsilon(-\sigma dz)^{(-1)^m \chi(M)} L_f(\sigma z)^{(-1)^m}\right)^{(-1)^r} \\ &= R_f(z)^{(-1)^m} \epsilon^{(-1)^r} (-\sigma dz)^{(-1)^{m+r} \chi(M)}. \end{aligned} \quad \square$$

### 3. THE REIDEMEISTER ZETA FUNCTION AND THE REIDEMEISTER TORSION OF THE MAPPING TORUS

Like the Euler characteristic, the Reidemeister torsion is algebraically defined.

Roughly speaking, the Euler characteristic is a graded version of the dimension, extending the dimension from a single vector space to a complex of vector spaces. In a similar way, the Reidemeister torsion is a graded version of the absolute value of the determinant of an isomorphism of vector spaces.

Let  $d^i : C^i \rightarrow C^{i+1}$  be a cochain complex  $C^*$  of finite dimensional vector spaces over  $\mathbb{C}$  with  $C^i = 0$  for  $i < 0$  and large  $i$ . If the cohomology  $H^i = 0$  for all  $i$  we say that  $C^*$  is *acyclic*. If one is given positive densities  $\Delta_i$  on  $C^i$  then the Reidemeister torsion  $\tau(C^*, \Delta_i) \in (0, \infty)$  for acyclic  $C^*$  is defined as follows:

**Definition 3.1.** Consider a chain contraction  $\delta^i : C^i \rightarrow C^{i-1}$  for acyclic  $C^*$ , i.e. a linear map such that  $d \circ \delta + \delta \circ d = \text{id}$ . Then  $d + \delta$  determines a map  $(d + \delta)_+ : C^+ := \bigoplus C^{2i} \rightarrow C^- := \bigoplus C^{2i+1}$  and a map  $(d + \delta)_- : C^- \rightarrow C^+$ . Since the map  $(d + \delta)^2 = \text{id} + \delta^2$  is unipotent,  $(d + \delta)_+$  must be an isomorphism. One defines  $\tau(C^*, \Delta_i) := |\det(d + \delta)_+|$ .

Reidemeister torsion is defined in the following geometric setting. Suppose  $K$  is a finite complex and  $E$  is a flat, finite dimensional, complex vector

bundle with base  $K$ . We recall that a flat vector bundle over  $K$  is essentially the same thing as a representation of  $\pi_1(K)$  when  $K$  is connected. If  $p \in K$  is a base point then one may move the fibre at  $p$  in a locally constant way around a loop in  $K$ . This defines an action of  $\pi_1(K)$  on the fibre  $E_p$  of  $E$  above  $p$ . We call this action the holonomy representation  $\rho : \pi \rightarrow GL(E_p)$ .

Conversely, given a representation  $\rho : \pi \rightarrow GL(V)$  of  $\pi$  on a finite dimensional complex vector space  $V$ , one may define a bundle  $E = E_\rho = (\tilde{K} \times V)/\pi$ . Here  $\tilde{K}$  is the universal cover of  $K$ , and  $\pi$  acts on  $\tilde{K}$  by covering transformations and on  $V$  by  $\rho$ . The holonomy of  $E_\rho$  is  $\rho$ , so the two constructions give an equivalence of flat bundles and representations of  $\pi$ .

If  $K$  is not connected then it is simpler to work with flat bundles. One then defines the holonomy as a representation of the direct sum of  $\pi_1$  of the components of  $K$ . In this way, the equivalence of flat bundles and representations is recovered.

Suppose now that one has on each fibre of  $E$  a positive density which is locally constant on  $K$ . In terms of  $\rho_E$  this assumption just means  $|\det \rho_E| = 1$ . Let  $V$  denote the fibre of  $E$ . Then the cochain complex  $C^i(K; E)$  with coefficients in  $E$  can be identified with the direct sum of copies of  $V$  associated to each  $i$ -cell  $\sigma$  of  $K$ . The identification is achieved by choosing a basepoint in each component of  $K$  and a basepoint from each  $i$ -cell. By choosing a flat density on  $E$  we obtain a preferred density  $\Delta_i$  on  $C^i(K, E)$ . A case of particular interest is when  $E$  is an acyclic bundle, meaning that the twisted cohomology of  $E$  is zero ( $H^i(K; E) = 0$ ). In this case one defines the Reidemeister torsion of  $(K, E)$  to be  $\tau(K; E) = \tau(C^*(K; E), \Delta_i) \in (0, \infty)$ . It does not depend on the choice of flat density on  $E$ .

The Reidemeister torsion of an acyclic bundle  $E$  on  $K$  has many nice properties. Suppose that  $A$  and  $B$  are subcomplexes of  $K$ . Then we have a multiplicative law:

$$(3) \quad \tau(A \cup B; E) \cdot \tau(A \cap B; E) = \tau(A; E) \cdot \tau(B; E)$$

that is interpreted as follows. If three of the bundles  $E|A \cup B$ ,  $E|A \cap B$ ,  $E|A$ ,  $E|B$  are acyclic then so is the fourth and the equation (3) holds.

Another property is the simple homotopy invariance of the Reidemeister torsion. Suppose  $K'$  is a subcomplex of  $K$  obtained by an elementary collapse of an  $n$ -cell  $\sigma$  in  $K$ . This means that  $K = K' \cup \sigma \cup \sigma'$  where  $\sigma'$  is an  $(n-1)$  cell of  $K$  so set up that  $\partial\sigma' = \sigma' \cap K'$  and  $\sigma' \subset \partial\sigma$ , i.e.  $\sigma'$  is a free face of  $\sigma$ . So one can push  $\sigma'$  through  $\sigma$  into  $K'$  giving a homotopy equivalence. Then

$$(4) \quad \tau(K; E) = \tau(K'; E)$$

By iterating a sequence of elementary collapses and their inverses, one obtains a homotopy equivalence of complexes that is called *simple*. Plainly one has, by iterating (4), that the Reidemeister torsion is a simply homotopy invariant. In particular  $\tau$  is invariant under subdivision. This implies

that for a smooth manifold, one can unambiguously define  $\tau(K; E)$  to be the torsion of any smooth triangulation of  $K$ .

In the case  $K = S^1$  is a circle, let  $A$  be the holonomy of a generator of the fundamental group  $\pi_1(S^1)$ . One has that  $E$  is acyclic if and only if  $I - A$  is invertible and then

$$(5) \quad \tau(S^1; E) = |\det(I - A)|$$

Note that the choice of generator is irrelevant as  $I - A^{-1} = (-A^{-1})(I - A)$  and  $|\det(-A^{-1})| = 1$ .

These three properties of the Reidemeister torsion are the analogues of the properties of Euler characteristic (cardinality law, homotopy invariance and normalization on a point), but there are differences. Since a point has no acyclic representations ( $H^0 \neq 0$ ) one cannot normalize  $\tau$  on a point as we do for the Euler characteristic, and so one must use  $S^1$  instead. The multiplicative cardinality law for the Reidemeister torsion can be made additive just by using  $\log \tau$ , so the difference here is inessential. More important for some purposes is that the Reidemeister torsion is not an invariant under a general homotopy equivalence: as mentioned earlier this is in fact why it was first invented.

It might be expected that the Reidemeister torsion counts something geometric (like the Euler characteristic). D. Fried [15] showed that it counts the periodic orbits of a flow and the periodic points of a map. We will show that the Reidemeister torsion counts the periodic point classes of a map (fixed point classes of the iterations of the map).

Some further properties of  $\tau$  describe its behavior under bundles. Let  $p : X \rightarrow B$  be a simplicial bundle with fiber  $F$  where  $F, B, X$  are finite complexes and  $p^{-1}$  sends subcomplexes of  $B$  to subcomplexes of  $X$  over the circle  $S^1$ . We assume here that  $E$  is a flat, complex vector bundle over  $B$ . We form its pullback  $p^*E$  over  $X$ . Note that the vector spaces  $H^i(p^{-1}(b), \mathbb{C})$  with  $b \in B$  form a flat vector bundle over  $B$ , which we denote  $H^i F$ . The integral lattice in  $H^i(p^{-1}(b), \mathbb{R})$  determines a flat density by the condition that the covolume of the lattice is 1. We suppose that the bundle  $E \otimes H^i F$  is acyclic for all  $i$ . Under these conditions D. Fried [15] has shown that the bundle  $p^*E$  is acyclic, and

$$(6) \quad \tau(X; p^*E) = \prod_i \tau(B; E \otimes H^i F)^{(-1)^i}.$$

the opposite extreme is when one has a bundle  $E$  on  $X$  for which the restriction  $E|_F$  is acyclic. Then, for  $B$  connected,

$$(7) \quad \tau(X; E) = \tau(F; E|_F)^{\chi(B)}$$

Suppose in (7) that  $F = S^1$ , i.e.,  $X$  is a circle bundle. Then (7) can be regarded as saying that

$$\log \tau(X; E) = \chi(B) \cdot \log \tau(F; E|_F)$$

is counting the circle fibers in  $X$  in the way that  $\chi$  counts points in  $B$ , with a weighting factor of  $\log \tau(F; E|F)$ .

Let  $f : K \rightarrow K$  be a homeomorphism of a compact polyhedron  $K$ . Let  $T_f := (K \times I)/(k, 0) \sim (f(k), 1)$  be the mapping torus of  $f$ .

We shall consider the bundle  $p : T_f \rightarrow S^1$  over the circle  $S^1$ . We assume here that  $E$  is a flat, complex vector bundle with finite dimensional fibre and base  $S^1$ . We form its pullback  $p^*E$  over  $T_f$ . Note that the vector spaces  $H^i(p^{-1}(b), \mathbb{C})$  with  $b \in S^1$  form a flat vector bundle over  $S^1$ , which we denote by  $H^i K$ . The integral lattice in  $H^i(p^{-1}(b), \mathbb{R})$  determines a flat density by the condition that the covolume of the lattice is 1. We suppose that the bundle  $E \otimes H^i K$  is acyclic for all  $i$ . Under these conditions from (6) it follows that the bundle  $p^*E$  is acyclic, and we have

$$(8) \quad \tau(T_f; p^*E) = \prod_i \tau(S^1; E \otimes H^i K)^{(-1)^i}.$$

Let  $g$  be the preferred generator of the group  $\pi_1(S^1)$  and let  $A = \rho(g)$  where  $\rho : \pi_1(S^1) \rightarrow GL(V)$ . Then the holonomy around  $g$  of the bundle  $E \otimes H^i K$  is  $A \otimes (f^*)^i$ . It follows from (5) and (8) that

$$(9) \quad \tau(T_f; p^*E) = \prod_i |\det(I - A \otimes (f^*)^i)|^{(-1)^i}.$$

We now consider the special case in which  $E$  is one-dimensional, so  $A$  is just a complex scalar  $\lambda$  of modulus one. Then in terms of the rational function  $L_f(z)$  we have :

$$(10) \quad \tau(T_f; p^*E) = \prod_i |\det(I - \lambda(f^*)^i)|^{(-1)^i} = |L_f(\lambda)|^{-1}$$

**Theorem 3.2.** *Let  $f : M \rightarrow M$  be a homeomorphism of an infra-nilmanifold  $M$ . Assume that  $N(f) = |L(f)|$ . Then*

$$\tau(T_f; p^*E) = |L_f(\lambda)|^{-1} = |N_f(\sigma\lambda)|^{(-1)^{r+1}} = |R_f(\sigma\lambda)|^{(-1)^{r+1}}$$

where  $\sigma = (-1)^p$ ,  $p$  is the number of real eigenvalues of  $F^*$  in the region  $(-\infty, -1)$  and  $r$  is the number of real eigenvalues of  $F^*$  whose absolute value is greater than 1.

*Proof.* From Theorem 1.6 it follows that  $N_f(z) = R_f(z) = (L_f(\sigma z))^{(-1)^r}$ . The theorem then follows from formula (10).  $\square$

#### 4. EXAMPLES

In this section we find some examples in which the Anosov relation does not hold, but the Reidemeister zeta function is a rational function or a root from rational function.

**Example 4.1.** This is an example used by Anosov to show that the Anosov relation does not hold when the manifold is not a nilmanifold [1].

Let  $\alpha = (a, A)$  and  $t_i = (e_i, I_2)$  be elements of  $\mathbb{R}^2 \rtimes \text{Aut}(\mathbb{R}^2)$ , where

$$a = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then  $A$  has period 2,  $(a, A)^2 = (a + Aa, I_2) = (e_1, I_2)$ , and  $t_2\alpha = \alpha t_2^{-1}$ . Let  $\Gamma$  be the subgroup generated by  $t_1$  and  $t_2$ . Then it forms a lattice in  $\mathbb{R}^2$  and the quotient space  $\Gamma \backslash \mathbb{R}^2$  is the 2-torus. It is easy to check that the subgroup

$$\Pi = \langle \Gamma, (a, A) \rangle \subset \mathbb{R}^2 \rtimes \text{Aut}(\mathbb{R}^2)$$

generated by the lattice  $\Gamma$  and the element  $(a, A)$  is discrete and torsion free. Furthermore,  $\Gamma$  is a normal subgroup of  $\Pi$  of index 2. Thus  $\Pi$  is an (almost) Bieberbach group, which is the Klein bottle group, and the quotient space  $\Pi \backslash \mathbb{R}^2$  is the Klein bottle. Thus  $\Gamma \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$  is a double covering projection.

Let  $K : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be the linear automorphism given by

$$K = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}.$$

It is not difficult to check that  $K$  induces  $\psi_K : \Gamma \backslash \mathbb{R}^2 \rightarrow \Gamma \backslash \mathbb{R}^2$  and  $\Psi_K : \Pi \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$  so that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{K} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \Gamma \backslash \mathbb{R}^2 & \xrightarrow{\psi_K} & \Gamma \backslash \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \Pi \backslash \mathbb{R}^2 & \xrightarrow{\Psi_K} & \Pi \backslash \mathbb{R}^2 \end{array}$$

Note that all the vertical maps are the natural covering maps. In particular,  $\Gamma \backslash \mathbb{R}^2 \rightarrow \Pi \backslash \mathbb{R}^2$  is a double covering by the holonomy group of  $\Pi/\Gamma$ , which is  $\Phi = \{I, A\} \cong \mathbb{Z}_2$ .

By Theorem 1.2, we have

$$\begin{aligned} L(\Psi_K^n) &= \frac{1}{2} (\det(I - K^n) + \det(I - AK^n)) = 1 - (-1)^n, \\ N(\Psi_K^n) &= 2^n (1 - (-1)^n). \end{aligned}$$

In particular,  $R(\Psi_K^n) = 2^{n+1}$  when  $n$  is odd; otherwise,  $R(\Psi_K^n) = \infty$ . Therefore, the Reidemeister zeta function  $R_{\Psi_K}(z)$  is not defined, and

$$\begin{aligned} L_{\Psi_K}(z) &= \exp \left( \sum_{n=1}^{\infty} \frac{2}{2n-1} z^{2n-1} \right) = \frac{1+z}{1-z}, \\ N_{\Psi_K}(z) &= \exp \left( \sum_{n=1}^{\infty} \frac{2^{2n}}{2n-1} z^{2n-1} \right) \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{2}{2n-1} (2z)^{2n-1} \right) = \frac{1+2z}{1-2z}. \end{aligned}$$

**Example 4.2.** Consider Example 3.10 of [20] in which an infra-nilmanifold  $M$  modeled on the 3-dimensional Heisenberg group  $\text{Nil}$  has the holonomy group of order 2 generated by  $A$  and a self-map  $f$  on  $M$  is induced by the automorphism  $D : \text{Nil} \rightarrow \text{Nil}$  given by

$$D : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & x+y & -z + \frac{1}{2}x^2 + xy \\ 0 & 1 & x \\ 0 & 0 & 1 \end{bmatrix}.$$

Then with respect to the ordered (linear) basis for the Lie algebra of  $\text{Nil}$

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the differentials of  $A$  and  $D$  are

$$A_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad D_* = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

By Lemma 1.5, we have

$$R_\phi(z) = \sqrt{\exp \left( \sum_{n=1}^{\infty} \frac{|\det(I - D_*^n)|}{n} z^n \right)} \sqrt{\exp \left( \sum_{n=1}^{\infty} \frac{|\det(A_* - D_*^n)|}{n} z^n \right)}.$$

However in this example a simple computation shows that  $\det(I - D_*^n) = -\det(A_* - D_*^n) \leq 0$  for all  $n$ . (Remark also that  $A_*$  is a block diagonal matrix with  $1 \times 1$  block  $I_1$  and  $2 \times 2$  block  $-I_2$ . Further,  $D_*$  is also a block diagonal matrix with  $1 \times 1$  block  $D_1$  and  $2 \times 2$  block  $D_2$ . We have

$$\begin{aligned} \det(A_* - D_*^n) &= \det(I_1 - D_1^n) \det(-I_2 - D_2^n) \\ &= \det(I_1 - D_1^n) \det(I_2 + D_2^n) \\ &= (1 - (-1)^n)(1 + \lambda_1^n)(1 + \lambda_2^n) \end{aligned}$$

and, similarly, we have

$$\begin{aligned}\det(I - D_*^n) &= \det(I_1 - D_1^n) \det(I_2 - D_2^n) \\ &= (1 - (-1)^n)(1 - \lambda_1^n)(1 - \lambda_2^n)\end{aligned}$$

where  $\lambda_i = \frac{1 \pm \sqrt{5}}{2}$  are the eigenvalues of  $D_2$ . We can see that  $(1 + \lambda_1^n)(1 + \lambda_2^n) + (1 - \lambda_1^n)(1 - \lambda_2^n) = 0$  for odd  $n$ . This proves our claim.)

Consequently, we obtain a rational function

$$\begin{aligned}N_f(z) = R_\phi(z) &= \exp \left( \sum_{n=1}^{\infty} \frac{|\det(I - D_*^n)|}{n} z^n \right) \\ &= \exp \left( - \sum_{n=1}^{\infty} \frac{\det(I - D_*^n)}{n} z^n \right) \\ &= \left( \prod_{i=0}^3 \det \left( I - \bigwedge^i z D_* \right)^{(-1)^{i+1}} \right)^{-1}.\end{aligned}$$

The self-maps on a flat manifold in the next example were considered in [2] as an illustration that the Nielsen zeta functions for infra-nilmanifolds are rational function. We compute these Nielsen zeta functions again using our own methods and we obtain the same rational functions!

**Example 4.3** ([2]). Consider  $\alpha = (a, U) \in \mathbb{R}^3 \rtimes \text{Aut}(\mathbb{R}^3)$  where

$$a = \begin{bmatrix} 0 \\ 0 \\ 1/2 \end{bmatrix}, \quad U = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then  $\alpha^2 = (a, U)^2 = (e_3, I)$ . Then  $(e_1, I), (e_2, I), \alpha$  generate a 3-dimensional Bieberbach group  $\Gamma$  with holonomy group  $\Phi = \langle U \rangle$ .

(1) The linear map  $D : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by the matrix

$$D = \begin{bmatrix} 4 & 2 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

induces a map  $f : \Gamma \backslash \mathbb{R}^3 \rightarrow \Gamma \backslash \mathbb{R}^3$ . Note that  $D$  has eigenvalues 2, 3 and 5.

Now we will evaluate  $N_f(z)$  using Lemma 1.5:

$$N_f(z) = \sqrt{\exp \left( \sum_{n=1}^{\infty} \frac{|\det(I - D^n)|}{n} z^n \right)} \sqrt{\exp \left( \sum_{n=1}^{\infty} \frac{|\det(U - D^n)|}{n} z^n \right)}.$$



Observe that

$$\begin{aligned}
|\det(U - D^n)| &= |\det(-I_2 - D_1^n) \det(I_1 - D_2^n)| \\
&= |\det(I_2 + D_1^n)| |\det(I_1 - D_2^n)| \\
&= (5^n - 1) \det(I_2 + D_1^n) \\
&= (5^n - 1) \sum_i \operatorname{tr}(\bigwedge^i D_1^n).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\exp \left( \sum_{n=1}^{\infty} \frac{|\det(U - D^n)|}{n} z^n \right) \\
&= \exp \left( \sum_{n=1}^{\infty} \frac{(5^n - 1) \sum_i \operatorname{tr}(\bigwedge^i D_1^n)}{n} z^n \right) \\
&= \exp \left( \sum_{n=1}^{\infty} \frac{\sum_i \operatorname{tr}(\bigwedge^i D_1^n)}{n} (5z)^n - \sum_{n=1}^{\infty} \frac{\sum_i \operatorname{tr}(\bigwedge^i D_1^n)}{n} z^n \right) \\
&= \prod_i \frac{\det(I - z \bigwedge^i D_1)}{\det(I - 5z \bigwedge^i D_1)} \\
&= \frac{1-z}{1-5z} \cdot \frac{1-5z+6z^2}{1-25z+150z^2} \cdot \frac{1-6z}{1-30z}
\end{aligned}$$

because  $\bigwedge^0 D_1 = 1$ ,  $\bigwedge^1 D_1 = D_1$ ,  $\bigwedge^2 D_1 = \det(D_1) = 6$ . In a similar fashion, we compute

$$\begin{aligned}
&\exp \left( \sum_{n=1}^{\infty} \frac{|\det(I - D_*^n)|}{n} z^n \right) \\
&= \exp \left( \sum_{n=1}^{\infty} \frac{|\det(I_2 - D_1^n) \det(I_1 - D_2^n)|}{n} z^n \right) \\
&= \exp \left( \sum_{n=1}^{\infty} \frac{(5^n - 1) \sum_i (-1)^i \operatorname{tr}(\bigwedge^i D_1^n)}{n} z^n \right) \\
&= \prod_i \left( \frac{\det(I - z \bigwedge^i D_1)}{\det(I - 5z \bigwedge^i D_1)} \right)^{(-1)^i} \\
&= \frac{1-z}{1-5z} \cdot \left( \frac{1-5z+6z^2}{1-25z+150z^2} \right)^{-1} \cdot \frac{1-6z}{1-30z}.
\end{aligned}$$

In all, we obtain that

$$N_f(z) = R_f(z) = \frac{1-z}{1-5z} \cdot \frac{1-6z}{1-30z}.$$

(2) The linear map  $E : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  defined by the matrix

$$E = \begin{bmatrix} -2 & 8 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

induces a map  $g : \Gamma \backslash \mathbb{R}^3 \rightarrow \Gamma \backslash \mathbb{R}^3$ . Note that  $E$  has eigenvalues 0, 2 and  $-3$ . Observe that

$$\begin{aligned} |\det(U - E^n)| &= |\det(-I_2 - E_1^n) \det(I_1 - E_2^n)| \\ &= |\det(I_2 + E_1^n)| |\det(I_1 - E_2^n)| \\ &= |1 - (-3)^n| \det(I_2 + E_1^n) \\ &= (3^n - (-1)^n) \sum_i \text{tr}(\bigwedge^i E_1^n). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} &\exp \left( \sum_{n=1}^{\infty} \frac{|\det(U - E^n)|}{n} z^n \right) \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{(3^n - (-1)^n) \sum_i \text{tr}(\bigwedge^i E_1^n)}{n} z^n \right) \\ &= \exp \left( \sum_{n=1}^{\infty} \frac{\sum_i \text{tr}(\bigwedge^i E_1^n)}{n} (3z)^n - \sum_{n=1}^{\infty} \frac{\sum_i \text{tr}(\bigwedge^i E_1^n)}{n} (-z)^n \right) \\ &= \prod_i \frac{\det(I + z \bigwedge^i E_1)}{\det(I - 3z \bigwedge^i E_1)} \\ &= \frac{1+z}{1-3z} \cdot \frac{1+2z}{1-6z} \end{aligned}$$

because  $\bigwedge^0 E_1 = 1$ ,  $\bigwedge^1 E_1 = E_1$ ,  $\bigwedge^2 E_1 = \det(E_1) = 0$ . In a similar fashion, we compute

$$\begin{aligned}
& \exp \left( \sum_{n=1}^{\infty} \frac{|\det(I - E_*^n)|}{n} z^n \right) \\
&= \exp \left( \sum_{n=1}^{\infty} \frac{|\det(I_2 - E_1^n) \det(I_1 - E_2^n)|}{n} z^n \right) \\
&= \exp \left( \sum_{n=1}^{\infty} \frac{-(3^n - (-1)^n) \sum_i (-1)^i \text{tr}(\bigwedge^i E_1^n)}{n} z^n \right) \\
&\quad (\det(I_2 - E_1^n) < 0 \ \forall n) \\
&= \prod_i \left( \frac{\det(I - 3z \bigwedge^i E_1)}{\det(I + z \bigwedge^i E_1)} \right)^{(-1)^i} \\
&= \frac{1 - 3z}{1 + z} \cdot \left( \frac{1 - 6z}{1 + 2z} \right)^{-1}.
\end{aligned}$$

In all, we obtain that

$$N_f(z) = R_f(z) = \frac{1 + 2z}{1 - 6z}.$$

**Example 4.4.** Consider Example 3.5 of [22] in which an infra-nilmanifold  $M$  modeled on the 3-dimensional Heisenberg group  $\text{Nil}$  has the holonomy group of order 2 generated by  $A$  and a self-map  $f$  on  $M$  is induced by the automorphism  $D : \text{Nil} \rightarrow \text{Nil}$  given by

$$D : \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \mapsto \begin{bmatrix} 1 & -4x - y & z' \\ 0 & 1 & 6x + 2y \\ 0 & 0 & 1 \end{bmatrix}$$

where  $z' = -2z - (12x^2 + 10xy + y^2)$ . Then with respect to the ordered (linear) basis for the Lie algebra of  $\text{Nil}$

$$\mathbf{e}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{e}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the differentials of  $A$  and  $D$  are

$$A_* = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad D_* = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -4 & -1 \\ 0 & 6 & 2 \end{bmatrix}.$$

By Lemma 1.5, we have

$$R_\phi(z) = \sqrt{\exp \left( \sum_{n=1}^{\infty} \frac{|\det(I - D_*^n)|}{n} z^n \right)} \sqrt{\exp \left( \sum_{n=1}^{\infty} \frac{|\det(A_* - D_*^n)|}{n} z^n \right)}.$$

Remark that  $A_*$  is a block diagonal matrix with  $1 \times 1$  block  $I_1$  and  $2 \times 2$  block  $-I_2$ . We have

$$\begin{aligned}
|\det(A_* - D_*^n)| &= |\det(I_1 - D_1^n) \det(-I_2 - D_2^n)| \\
&= |\det(I_1 - D_1^n)| |\det(I_2 + D_2^n)| \\
&= |(1 - (-2)^n)| (-1)^n \det(I_2 + D_2^n) \\
&= (2^n - (-1)^n) (-1)^n \sum_i \text{tr}(\bigwedge^i D_2^n).
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\exp\left(\sum_{n=1}^{\infty} \frac{|\det(A_* - D_*^n)|}{n} z^n\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(2^n - (-1)^n) (-1)^n \sum_i \text{tr}(\bigwedge^i D_2^n)}{n} z^n\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{\sum_i \text{tr}(\bigwedge^i D_2^n)}{n} (-2z)^n - \sum_{n=1}^{\infty} \frac{\sum_i \text{tr}(\bigwedge^i D_2^n)}{n} z^n\right) \\
&= \prod_i \frac{\det(I - z \bigwedge^i D_2)}{\det(I + 2z \bigwedge^i D_2)} \\
&= \frac{1-z}{1+2z} \cdot \frac{1+2z-2z^2}{1-4z-8z^2} \cdot \frac{1+2z}{1-4z}.
\end{aligned}$$

In a similar fashion, we compute

$$\begin{aligned}
&\exp\left(\sum_{n=1}^{\infty} \frac{|\det(I - D_*^n)|}{n} z^n\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{|\det(I_1 - D_1^n) \det(I_2 - D_2^n)|}{n} z^n\right) \\
&= \exp\left(\sum_{n=1}^{\infty} \frac{(2^n - (-1)^n) (-1)^{n+1} \sum_i (-1)^i \text{tr}(\bigwedge^i D_2^n)}{n} z^n\right) \\
&= \prod_i \left( \frac{\det(I + 2z \bigwedge^i D_2)}{\det(I - z \bigwedge^i D_2)} \right)^{(-1)^i} \\
&= \frac{1+2z}{1-z} \cdot \frac{1+2z-2z^2}{1-4z-8z^2} \cdot \frac{1-4z}{1+2z}.
\end{aligned}$$

The last identity of the above computations follows from the definition of  $\bigwedge^i D_2$  (cf. see [16, Lemma 3.2]). Namely, we have

$$\bigwedge^0 D_2 = 1, \quad \bigwedge^1 D_2 = D_2, \quad \bigwedge^2 D_2 = \det(D_2) = -2.$$

In all, we obtain that

$$N_f(z) = R_\phi(z) = \frac{1 + 2z - 2z^2}{1 - 4z - 8z^2}.$$

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